

An approach to F_1 via the theory of Λ -rings

by Stanislaw Betley

0. Introduction.

This note is devoted to the preliminary study of the concept of Borger from [B], that the decent data from \mathbf{Z} to F_1 for commutative rings is the Λ -structure. More precisely, he claims, that the category of rings over F_1 should consist of Λ -rings and the restriction of scalars from \mathbf{Z} to F_1 takes any commutative ring R to its ring of Witt vectors $W(R)$ with the canonical Λ -structure. In this approach the mythical field F_1 is equal to the ring of integers \mathbf{Z} with the canonical Λ -structure. We will denote it as \mathbf{Z}_λ or just F_1 throughout the sections 1-4.

In [Be] we proved that the categorical ζ -function of the category of commutative monoids calculates the Riemann ζ -function of the integers. This was done in order to support the idea, that while trying to view \mathbf{Z} as a variety over the field with one element we should consider integers as multiplicative monoid without addition. The idea that \mathbf{Z} treated as a variety over F_1 should live in the category of monoids is well described in the literature, see for example [KOW] or [D]. But, because category of monoids is too rigid, most authors instead of working with monoids directly extend their field of scalars from F_1 to \mathbf{Z} (or other rings), assuming that scalar extension from F_1 to \mathbf{Z} should take a monoid A to its monoid ring $\mathbf{Z}[A]$. This agrees well with the expectation that rings should be treated in the category of monoids as monoids with ring multiplication as a monoidal operation. Then the forgetful functor from rings to monoids and the scalar extension as described above give us the nice pair of adjoint functors. But this approach carries one disadvantage. It takes us quickly from formally new approach via monoids to the classical world of rings and modules over them or to other abelian categories. Working with Λ -rings seems to be a good way of overcoming this disadvantage. We will preserve monoidal point of view but our approach will allow to use more algebra-geometric methods then the ones which are at hand in the world of monoids.

The definition of F_1 as above does describe this object almost like a field. If Λ -operations are part of the structure then the ideals in our rings should be preserved by them. It is easy to observe that in \mathbf{Z}_λ there are no proper ideals preserved by Λ -operations. We will try to justify the following point of view that in the category of Λ -rings we can calculate the Riemann ζ of the integers in two ways. One - as the ζ -function of the category of modules over \mathbf{Z}_λ and the other (geometric) via calculating numbers of fixed points of the action of the Frobenius morphism on the affine line over the algebraic closure of \mathbf{Z}_λ .

In order to show that our program works we will try first of all to answer the question what is the algebraic closure of F_1 . Then we show that the ζ -function of the affine line over F_1 calculated by the generating function is the same as the ζ -function of integers. Next we have to find the proper category of modules over F_1 and calculate its categorical

ζ function. At the end we prove that the categorical ζ -function of the category of modules over F_1 is equal to the ζ -function of the integers.

I. F_1 and the category of commutative monoids.

Let M_{ab} denote the category of commutative monoids with unit and unital maps. This section is devoted only to the rough description of certain features of M_{ab} which are crucial for our approach in the next sections. They are either obvious or well described elsewhere so we are very brief here.

Let us start from a some piece of notation. We will denote by \mathbf{N}^* the multiplicative monoid of natural numbers. The symbol \mathbf{N}^+ denotes the monoid of natural numbers with addition. Observe that any monoid $M \in M_{ab}$ carries natural action of \mathbf{N}^* by identifying $k \in \mathbf{N}^*$ with $\psi^k : M \rightarrow M$ where $\psi^k(m) = m^k$. This structure is obvious in M_{ab} and adds very little while studying monoids, but should be reflected always, when we want to induce structures from M_{ab} to other (abelian) categories. This action will be addressed as an action of Adams operations on M . We can also interpret it as an action of the powers of the Frobenius endomorphism. Observe that in characteristic p the Frobenius endomorphism acts by rising an element to the p th power. We propose to view the Frobenius action in a uniform, characteristic free way. In any structure by Frobenius action we mean an action of \mathbf{N}^* where $k \in \mathbf{N}^*$ acts by rising an element to the k th power if this gives a morphism in the considered structure or k acts as identity.

In [D] Deitmar developed the algebraic geometry in the setting of monoids, associating to $M \in Obj(M_{ab})$ a topological space with a sheaf of monoids which resembles a spectrum of prime ideals of a commutative ring with its structural sheaf. In his language F_1 is the one element monoid consisting only of the unit.

Definition 1.1: For any unital monoid M we define the polynomial ring over M by

$$M[X] = M \times \mathbf{N}^+.$$

We write ax^n for the element (a, n) with convention $(a, 0) = a$. The evaluation of ax^n at the point $b \in M$ equals ab^n .

Remark 1.2: Our definition agrees with the general expectation (compare [D], [S]) that scalar extension from F_1 to \mathbf{Z} should take any monoid A to its integral monoid ring $\mathbf{Z}[A]$. Indeed we have an isomorphism of rings

$$\mathbf{Z}[M \times \mathbf{N}^+] \rightarrow \mathbf{Z}[M][X]$$

given by the formula

$$\sum_{i \in I} z_i(a_i, n_i) \mapsto \sum z_i a_i X^{n_i}$$

Since we know what is the polynomial ring over a monoid we can try to imagine what is the algebraic closure of F_1 . Typically, when we want to close algebraically a field K we

have to add to it, at least, all roots of elements of K (this is sufficient in the finite field case). In other words we can say that \bar{K} is the minimal field which contains all roots of the elements of K and has no algebraic extensions. This second condition is equivalent to saying that all roots of elements of \bar{K} are contained in \bar{K} . So it is not surprising to say:

Remark 1.3: $\bar{F}_1 = \mathbf{Q}/\mathbf{Z}$

For any monoid M we write M_+ for M with 0 added to it. This is important when we want to talk about geometric points of varieties over F_1 . In [D] prime ideals in M are given by subsets of M satisfying obvious for ideals conditions. Empty set is a good prime ideal, and it is necessary to consider it at least in order to have one point in $\text{Spec}(F_1)$. When we have a variety M over F_1 (a monoid) and B is an extension of F_1 we can talk about $M(B)$, the B -points of M . They are equal (following [D]) to $\text{Hom}_{M_{ab}}(M_+, B_+)$. Here we have to add zeros to monoids because a map cannot have an empty value on some element.

We can calculate the number of points of \mathbf{N}^+ , treated as an affine line over F_1 , in \bar{F}_1 . The Frobenius action on \mathbf{Q}/\mathbf{Z} extends to the action on \mathbf{Q}/\mathbf{Z} -points of \mathbf{N}^+ . We easily calculate that for $k \in \mathbf{N}^*$ we have the following formula for the order of the set of fixed points:

$$|(\mathbf{N}^+(\mathbf{Q}/\mathbf{Z}))^k| = k$$

If we view the Weil zeta function for varieties over F_p as a way of keeping track of the numbers of their points over \bar{F}_p fixed by the action of the powers of the Frobenius morphisms then we get the formula for the Riemann zeta function ζ_R in the similar spirit in M_{ab} :

$$\zeta(\mathbf{N}^+, s) = \sum_{k=1}^{\infty} |(\mathbf{N}^+(\mathbf{Q}/\mathbf{Z}))^k|^{-s}$$

As we said in the introduction the geometry in M_{ab} is too weak to expect that we can fulfill Deligne's program there and approach the Riemann conjecture. But the picture described above is very basic and leads to the correct ζ function. This implies that we want to see our more sophisticated structures of F_1 -objects as the structures induced from the picture described above.

II. Preliminaries on Λ -rings.

Our rings are always commutative with units. Following [Y, Definition 1.10] we have:

Definition 2.1: A λ -ring is a ring R together with functions

$$\lambda^n : R \rightarrow R \quad (n \geq 0)$$

satisfying for any $x, y \in R$:

- (1) $\lambda^0(x) = 1$,
- (2) $\lambda^1(x) = x$,
- (3) $\lambda^n(1) = 0$ for $n \geq 2$,

- (4) $\lambda^n(x + y) = \sum_{i+j=n} \lambda^i(x) \lambda^j(y)$,
- (5) $\lambda^n(xy) = P_n(\lambda^1(x), \dots, \lambda^n(x); \lambda^1(y), \dots, \lambda^n(y))$,
- (6) $\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), \dots, \lambda^{nm}(x))$.

Above P_n and $P_{n,m}$ are certain universal polynomials with integer coefficients obtained via symmetric functions theory (see [Y, Example 1.9 and 1.7]). By a homomorphism of Λ -rings we mean a ring homomorphism which commutes with Λ -operations. We say that $x \in R$ is of degree k , if k is the largest integer for which $\lambda^k(x) \neq 0$. If such finite k does not exist we say that x is of infinite degree. Observe that (by formula 4) the map

$$R \ni x \mapsto \sum_{i \geq 0} \lambda^i(x) t^i$$

is a homomorphism from the additive group of R to the multiplicative group of power series over R with constant term 1. We will denote this map as $\lambda_t(x)$. Observe also that $\lambda_t(0) = 1$ and hence $\lambda_t(-r) = \lambda_t(r)^{-1}$.

Ring of integers \mathbf{Z} carries the unique, canonical Λ -ring structure described by the formula $\lambda^n(m) = \binom{m}{n}$. Similarly all integral monoid rings $\mathbf{Z}[M]$ will be considered with the Λ structure defined for any $m \in M$ by formulas

$$\lambda^1(m) = m$$

$$\lambda^i(m) = 0 \quad \text{for } i > 1.$$

We will always considered integral monoid rings with such Λ -structure, because it corresponds well with the monoidal point of view on the field with one elements.

Lemma 2.2: Let R be equal to the monoidal Λ -ring $\mathbf{Z}[M]$ with the Λ -structure defined above. Then in R only generators $m \in M \subset \mathbf{Z}[M]$ are of degree 1.

Proof. We know that the elements of M are of degree 1. By separating positive and negative coefficients we get $\mathbf{Z}[M] \ni r = \sum_{i=1}^k a_i m_i - \sum_{j=1}^l b_j m_j$, where all a_i s and b_j s belong to \mathbf{Z} and are greater than 0. Observe that the assumption that r is of degree 1 implies that $\lambda_t(r) = 1 + bt$ for a certain $b \in \mathbf{Z}[M]$. We can easily calculate λ_t -functions in the case of monoidal rings. Let $m \in M$ and a be a positive integer. Then

$$\lambda_t(am) = \lambda_t(m)^a = (1 + mt)^a$$

Hence we easily get

$$\lambda_t\left(\sum_{i=1}^k a_i m_i - \sum_{j=1}^l b_j m_j\right) = \prod_{i=1}^k (1 + m_i t)^{a_i} / \prod_{j=1}^l (1 + m_j t)^{b_j}$$

If r is of degree 1 we have equality

$$(*) \quad \prod_{i=1}^k (1 + m_i t)^{a_i} = (1 + bt) \prod_{j=1}^l (1 + m_j t)^{b_j}$$

From this, by comparing coefficients at the highest degree of t we get

$$b = \prod_{i=1}^k (m_i)^{a_i} / \prod_{j=1}^l (m_j)^{b_j}$$

and hence $b \in M$. On the other hand, when we calculate the coefficient at the first degree in the equality (*) we get

$$\sum_{i=1}^k a_i m_i = b + \sum_{j=1}^l b_j m_j$$

But by the definition of a_i s and b_j s this is possible only when $r = b \in M$.

Let M_{ab} denote the category of commutative monoids with unit and $Ring^\lambda$ stands for the category of commutative unital rings with Λ -structure. We have:

Proposition 2.3: The functor $M_{ab} \rightarrow Ring^\lambda$, which takes a monoid M to the Λ -ring $\mathbf{Z}[M]$ with the Λ -structure defined above has a right adjoint $Ring^\lambda \rightarrow M_{ab}$ which takes a Λ -ring R to the multiplicative monoid R_1 of its elements of degree not exceeding 1.

Proof. By [Y, Proposition 1.13] we know that in any Λ -ring the product of 1-dimensional elements is again 1-dimensional (or equal to 0). Hence R_1 is a well defined multiplicative submonoid of R considered here as the multiplicative monoid. If $f \in Mor_{Ring^\lambda}(R, S)$ then f carries 1 dimensional elements of R to 1 dimensional elements of S or to 0 by the definition of a Λ -homomorphism. Hence our right adjoint is well defined. The rest of the proof is obvious.

The Λ -operations on a ring R define on it the sequence of Adams operations $\psi^k : R \rightarrow R$ which are natural ring homomorphisms. They can be defined by the Newton formula:

$$\psi^k(x) - \lambda^1(x)\psi^{k-1}(x) + \dots + (-1)^{k-1}\lambda^{k-1}(x)\psi^1(x) = (-1)^{k-1}k\lambda^k(x)$$

For their properties see [Y, chapter 3]. It is straightforward to check that the canonical λ -structure on \mathbf{Z} defines trivial Adams operations and the formula $m \mapsto m^k$ for $m \in M$ determines the k th Adams operation on the monoidal ring $\mathbf{Z}[M]$. Adams operations can be viewed always as an action on a considered structure by the multiplicative monoid \mathbf{N}^* of natural numbers. Every object M of M_{ab} has naturally such a structure as was described in Section 1. So proposition 2.3 can be viewed as a statement about adjoint functors between categories with objects carrying the action of \mathbf{N}^* . It is easy to check that the \mathbf{N}^* action on $\mathbf{Z}[M]$ given by $k(m) = m^k$ while treated as the action of Adams operations forces to have λ -structure on $\mathbf{Z}[M]$ satisfying $\lambda^i(m) = 0$ for $i > 1$.

Observe that if we consider natural numbers \mathbf{N}^+ as a monoid with addition, then for any ring R we have a description of the polynomial ring over R via the formula

$$R[x] = R[\mathbf{N}^+] = R \otimes \mathbf{Z}[\mathbf{N}^+].$$

Hence we will consider $\mathbf{Z}[\mathbf{N}^+]$ as polynomial ring over F_1 in the rest of the paper, with Λ -structure defined like for any other monoidal ring. Moreover, for any Λ -ring R we

have well defined Λ -structure on $R[x]$ because tensor product of rings inherits it from the Λ -structures of the factors.

Definition 2.4: Let R be a Λ -ring and I is an ideal in R . We will call it a Λ -ideal if it is preserved under the action of λ^k , for any $k > 0$.

It is straightforward to check that if we divide a Λ -ring by a Λ -ideal then R/I carries the induced Λ -structure and the quotient homomorphism $R \rightarrow R/I$ is a homomorphism of Λ -rings. Of course the opposite is also true: a kernel of the Λ -rings homomorphism is a Λ -ideal. For computations important is that an ideal I in a Λ -ring R with \mathbf{Z} -torsion free quotient is a Λ -ideal if and only if it is preserved by the Adams operations (see [Y, Corollary 3.16]).

III. Algebraic extensions of F_1 and its algebraic closure \bar{F}_1 .

As we said in the introduction, the Λ -ring \mathbf{Z}_λ (our hypothetical F_1) can be treated as a field because it contains no proper Λ -ideals. We have defined the ring of polynomials over F_1 . On the other hand we can always view an algebraic closure of a field via the ring of polynomials and its quotients. If K is a field we know that every algebraic extension of K should be contained in \bar{K} . We know that every algebraic extension of K is build out of simple extensions $K \subset K(a)$ where a is a root of a non-decomposable polynomial $f_a \in K[x]$. Moreover, for simple extensions we have the formula $K(a) = K[x]/(f_a)$. It means that we can view \bar{K} as a sum of the fields of the form $L[x]/(f)$ where L is an algebraic extension of K or even as a sum of simple extensions $K[x]/(f)$. We will try to use this point of view in our context but remembering all the time about the Λ -structures on our objects. Let us start from the following lemma:

Lemma 3.1: Every monic polynomial f , which generates the principal Λ -ideal in $F_1[x]$ has only roots of unity or 0 as his roots in \mathbf{C} . If $f(\mu_n) = 0$, where μ_n is the prime root of 1 of degree n , then $x^n - 1 \mid f$ (and hence $n \leq \deg(f)$).

Proof. Because our polynomials are monic the quotients of $F_1[x]$ by ideals generated by them are torsion free. Hence instead of working with λ -operations we can assume that our ideals are preserved by all Adams operations. Assume that an ideal I is generated by a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$. Then $\psi^k(a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) = a_n x^{kn} + a_{n-1} x^{k(n-1)} + \dots + a_0$. This formula implies that if a is a root of f then a^k is also a root of f for any natural k , when we calculate roots in the field of complex numbers. To see this observe that the evaluation at a map $e_a : F_1[x] \rightarrow \mathbf{C}$ is a homomorphism of rings. Let $I = \ker(e_a)$ then $(f) \subset I$ and because $\psi^k((f)) \subset (f)$ then $\psi^k(f) \in (f) \subset I$. But for a polynomial f the statement that if a is a root of f then a^k is also a root for any natural k implies directly that a is a root of 1 or $a = 0$. Hence all the zeros of f are the roots of unity or are equal to 0. On the other hand if a is a primitive root of 1 of degree s and $f(a) = 0$ then all the other roots of unity of degree s are among the zeros of f , again by the argument with Adams operations. This implies immediately our statement.

Definition 3.2: Let K be a Λ -ring. We will say that $f \in K[x]$ is non-decomposable if the principal ideal (f) is preserved by the Λ -operations and there is no decomposition $f = f_1 \cdot f_2$ such that $\deg(f_1) > 0$, $\deg(f_2) > 0$ and both ideals (f_1) and (f_2) are preserved by the Λ -operations.

Lemma 3.1 describes what are the non-decomposable polynomials in $F_1[x]$. In our approach we would like to take the following definition for an algebraic extension in the category of Λ -rings (field extensions of F_1).

Definition 3.3: Let F be a Λ -ring without nilpotent elements. We say that:

- i. $K \supset F$ is a simple algebraic extension of F if K is isomorphic to $F[x]/(f)$ for some non-decomposable $f \in F[x]$ and K has no nontrivial Λ -quotients except F itself.
- ii. K is an algebraic extension of F of finite degree if there is a sequence of simple extensions $F \subset K_1 \subset \dots \subset K_s \subset K$.
- iii. K is an algebraic extension of F if every $k \in K$ is contained in a finite degree algebraic extension L_k of F , which is contained in K .

We say that L is an algebraic closure of K , $L = \bar{K}$, if L is an algebraic extension of K , for any finite degree algebraic extension $K \hookrightarrow L_1$ we have a factorization $K \hookrightarrow L_1 \hookrightarrow L$ and L is minimal with respect to this property. We care only about monic f 's because otherwise $F_1[x]/(f)$ contains torsion elements which are always nilpotent in any Λ -structure. Some explanation why we have to accept that a field has a nontrivial quotient is contained in Remark 3.5.

Proposition 3.4: $\bar{F}_1 = F_1[\mathbf{Q}/\mathbf{Z}]$.

Proof. Observe first that the simple algebraic extension of F_1 is isomorphic to $F_1[x]/(x^p - 1)$ for some prime number p . This follows directly from 3.1 because if our extension is of positive degree and has presentation as $F_1[x]/(f)$ then 3.1 implies that $x^k - 1$ divides f for a certain positive k . But then either our extension has the forbidden quotient $F_1[x]/(x^k - 1)$ (we will say in the future that f has a forbidden factor) or $f = x^p - 1$ for a prime p . So we see that every simple algebraic extension of F_1 is isomorphic to $F_1[\mu_p]$ for a prime number p which can also be interpreted as a group ring $\mathbf{Z}[C_p]$ for a cyclic group C_p of order p .

Now assume that K is a simple algebraic extension of $F_1[\mu_n]$. We know that K has a presentation as $F_1[\mu_n][x]/(f)$ for a certain non-decomposable $f \in F_1[\mu_n][x]$. Observe, that for any $a \in F_1[\mu_n]$, $\psi^n(a) \in \mathbf{Z}$. Moreover for any $g \in F_1[\mu_n][x]$, $\psi^n(g) \in F_1[x]$ by the description of Adams operations. We are using here first of all the fact that the $F_1[\mu_n][x] = F_1[\mu_n] \otimes_{\mathbf{Z}} F_1[x]$ and the Λ -operations on the tensor product are induced from those on the factors. Secondly, Adams operations are fully described by Λ -operations and they are natural ring homomorphisms.

So we know that $\psi^n(f) \in F_1[x]$. We can apply to it the same procedure as in the proof of 3.1 and get that if a is a root of $\psi^n(f)$ then all its powers have also this property. So any root of $\psi^n(f)$ is a root of 1 or 0 and the statement of 3.1 is fulfilled for $\psi^n(f)$. Because $f | \psi^n(f)$ we know that all roots of f are roots of 1 of degree not exceeding nk ,

where $k = \deg(f)$ (zero can be excluded from our considerations). The polynomial f cannot have any root of 1 of degree n . If this would be the case then, because $x^n - 1$ is fully decomposable in $F_1[\mu_n][x]$, f would have a forbidden quotient. Consider first the special case and assume that f has a root of prime degree p , and $(n, p) = 1$. This implies that $x^p - 1 \mid \psi^n(f)$. But $f \mid \psi^n(f)$ and f and $x^p - 1$ have a common root. Moreover $x^p - 1$ has no root besides 1 in $F_1[\mu_n]$. Hence $x^p - 1 \mid f$ and this implies that $f = x^p - 1$ or f has forbidden quotient. Now we can consider the general case. Let a be a root of f . We know that $a^{nk} = 1$ and $a^n \neq 1$. This implies that a is a root of 1 of degree ns and $s \mid k$. So $a^{ns} = (a^s)^n = 1$ and hence a^s is a root of 1 of degree n . By the same considerations as in the special case we get now that $f = x^p - \mu_n^i \in F_1[\mu_n][x]$ and $p = s$ is a prime.

Observe that with n and p as in the special case we have

$$F_1[\mu_n][x]/(x^p - 1) = \mathbf{Z}[C_n][C_p] = \mathbf{Z}[C_n \times C_p] = \mathbf{Z}[C_{np}] = F_1[\mu_{np}]$$

In the general case observe that $a\mu_n$ is of degree pn and freely generates $F_1[\mu_n][x]/(x^p - \mu_n^i)$ over \mathbf{Z} so we can write:

$$F_1[\mu_n][x]/(x^p - \mu_n^i) = \mathbf{Z}[C_{np}] = F_1[\mu_{np}]$$

This implies directly our proposition.

Remark 3.5: Unlike F_1 all field extensions of F_1 have nontrivial quotients so they contain "ideals". Obviously $(x - 1) \mid (x^n - 1)$ and this implies that there is a λ -ring map $F_1[\mu_n] \rightarrow F_1$ given by augmentation. This is caused by the fact that we induce our structures from the category M_{ab} where one point monoid $\mathbf{1}$ is a zero object. It means that beside the map $\mathbf{1} \rightarrow M$ for any $M \in M_{ab}$ we have also $M \rightarrow \mathbf{1}$ and this should be reflected in the category of λ -rings. Moreover in M_{ab} we have a notion of a quotient object, the quotient map there commutes always with the action of \mathbf{N}^* so we have to allow the existence of quotients of our "fields".

Since we have the algebraic closure of F_1 we can look for the "powers of the Frobenius morphism". Recall the well known fact from algebraic geometry over finite fields. If \bar{F}_p is an algebraic closure of F_p and X is a variety over F_p then

$$X(F_{p^n}) = X(\bar{F}_p)^{\mathbf{n}}$$

Here $X(K)$ denotes the K -points of X and $Y^{\mathbf{n}}$ denotes the fixed points of the n th power of the Frobenius morphism action on Y . We would like to have the similar formula over F_1 . For the points of the variety over different fields we have universal solution. If R is a Λ -ring and X is a variety over F_1 then

$$X(R) = \text{Mor}(R, X)$$

where the morphism set is taken in the category of varieties. In the affine case, but what we mean X is described by another Λ -ring S ($X = \text{spec}(S)$) we have as usual

$$\text{Mor}(R, X) = \text{Hom}_{\Lambda\text{-rings}}(S, R)$$

Affine case $R = \mathbf{Z}[M]$ with M - monoid is crucial for us and in this case we can give precise meaning to the superscript \mathbf{n} . A Λ -ring structure on R implies that there is a homomorphism $i : R \rightarrow W(R)$, where $W(R)$ is a ring of big Witt vectors over R . For any n we have a Frobenius morphism $f_n : W(R) \rightarrow W(R)$. If we view $W(R)$ as invertible formal power series in indeterminate t with addition given by series multiplication and multiplication coming from ghost coordinates then $i(r) = 1 + rt$ for any $r \in R$ of degree one. For such elements

$$f_n(1 + rt) = (1 + r^n t) = \psi^n(1 + rt)$$

Hence we can assume that Frobenius action of $n \in N$ on $\mathbf{Z}[M]$ is realized by rising monoidal generators to the n th power. In other words it means this action is realized by the Adams operations.

Let now $X = \mathbf{Z}[x]$. Then $X(\bar{F}_1) = \text{Hom}_{\Lambda\text{-rings}}(\mathbf{Z}[x], F_1[\mathbf{Q}/\mathbf{Z}])$. Observe that every Λ -homomorphism from $\mathbf{Z}[x]$ is determined by the image of x which should be contained in elements of degree not exceeding 1. In the case of monoidal rings

$$\text{Hom}_{\Lambda\text{-rings}}(\mathbf{Z}[x], \mathbf{Z}[M]) = M_+$$

because as an image of x can be taken any element of M or 0 and by 2.2 that is all. As usual $M_+ = M \cup \{0\}$. The Frobenius action is realized by Adams operations. Hence we calculate that the set $X(\bar{F}_1)^{\mathbf{n}}$ consists of the roots of unity of degree $n - 1$ plus additional element 0 so has cardinality n . From this we get the following formula for the ordinary Riemann ζ -function ζ_R :

$$\zeta_R = \sum_{n=1}^{\infty} 1/X(\bar{F}_1)^{\mathbf{n}}$$

where X is an affine line over F_1 equal to $\mathbf{Z}[x]$.

IV. Categorical ζ function over F_1 .

Let us start from recalling after Kurokawa (compare [K]) the definition of the zeta function of a category with 0. If \mathcal{C} is a category with 0 we say that $X \in \text{Ob}(\mathcal{C})$ is simple if for any object Y the set $\text{Hom}_{\mathcal{C}}(X, Y)$ consists only of monomorphisms and 0. Let $N(X)$, the norm of X , denote the cardinality of the set $\text{End}_{\mathcal{C}}(X)$. We say that an object X is finite if $N(X)$ is finite. We denote by $P(\mathcal{C})$ the isomorphism classes of all finite simple objects of \mathcal{C} . Then we define the zeta function of \mathcal{C} as

$$\zeta(s, \mathcal{C}) = \prod_{P \in P(\mathcal{C})} (1 - N(P)^{-s})^{-1}$$

In [K] Kurokawa studied the properties of such zeta functions but for us the crucial is:

Remark 4.1: Let Ab denote the category of abelian groups and ζ_R stands for the Riemann zeta function of the integers. Then

$$\zeta(s, Ab) = \zeta_R$$

The following observation was the starting point for our considerations and it underlines the role of the category of abelian monoids. Recall that M_{ab} denote the category of abelian monoids with unit and unital maps. In [Be] we proved:

Theorem 4.2:

$$\zeta_R \cdot (1 - 2^{-s})^{-1} = \zeta(s, M_{ab})$$

Hence the category of monoids carries all the information needed for calculating the Riemann ζ -function of the integers. We want to look at its categorical calculation from 4.1 in a slightly different way, which is suitable for generalizations. First of all we underline that we are working in the category of \mathbf{Z} -modules. There are good analogs of the category of modules over an object X of an abstract category \mathcal{C} which has 0 and all finite limits. Beck in [Bec] defined them as abelian group objects in the category of objects over X (see also [H, chapter 2]). As is shown in [H] the category of abelian group objects in the category of rings over a given ring R is equivalent to the category of R -modules, where an R -module X defines the square zero extension of R with X as a square-zero ideal. In the case of $R = \mathbf{Z}$ we get, as expected, the category of abelian groups. An abelian group X corresponds to the square zero extension $\mathbf{Z} \triangleright X$. The finite simple objects in the category of rings over \mathbf{Z} are easily seen to come from the simple abelian groups (finite cyclic groups C_p of prime order p). For a given p we see that $N(C_p)$ is equal to the cardinality of the set $Hom_{Rings/\mathbf{Z}}(\mathbf{Z}[x], \mathbf{Z} \triangleright C_p)$. The polynomial ring $\mathbf{Z}[x]$ is treated as a ring over \mathbf{Z} via the map which takes x to 0. All this means that we have the geometrical method for calculating the categorical ζ of integers. We just have to count the $\mathbf{Z} \triangleright C_p$ -points of the affine line over \mathbf{Z} .

We can perform the calculation as above for any commutative ring R because finite simple objects in the category of R -modules correspond to the maximal ideals $I \subset R$ with finite quotient and one checks immediately that the cardinality of $Hom_{Rings/R}(R[x], R \triangleright R/I)$ is the same as the cardinality of $Hom_{R-mod}(R/I, R/I)$. Moreover this cardinality is the same as the number of elements of the residue field at the closed point corresponding to I in $Spec(R)$ so we are really calculating the classical ζ -function of an affine variety $Spec(R)$.

Observe that we can perform the same calculations in the category M_{ab} , where the role of integers is played by the field of one element in the sense of [D]. But this gives us no new insight because in the world of monoids the field of one element is represented by one point monoid $\mathbf{1}$ consisting of 1 only, so we have equality of categories $M_{ab}/\mathbf{1} = M_{ab}$. The affine line over $\mathbf{1}$ is equal to the monoid of natural numbers with addition.

We want to promote here the point of view that calculations of categorical ζ function should have geometric meaning. By this we mean that if in a category \mathcal{C} we have "ground field A " related to it such that all objects in \mathcal{C} have the A -structure and moreover we know what is the affine line $A[x]$ in \mathcal{C} then categorical ζ -function of \mathcal{C} should be defined in a geometrical way. But of course in such generality we can expect to have artificial objects in \mathcal{C} . To exclude them we will call a finite simple object P in \mathcal{C} geometrically finite if $n(P) = |Mor_{\mathcal{C}}(A[x], P)|$ is finite. Then we define the geometrical ζ -function of \mathcal{C} by the formula

$$\zeta_g(s, \mathcal{C}) = \prod_{P \in P'(\mathcal{C})} (1 - n(P)^{-s})^{-1}$$

where $P'(C)$ denote the set of isomorphism classes of geometrically finite objects of \mathcal{C} with $n(P) > 1$. Geometrically finite simple object P with $n(P) > 1$ will be called as non-degenerate objects of \mathcal{C} . These are the only objects which are meaningful for calculating the geometrical Riemann ζ -function of \mathcal{C} . Observe that in M_{ab} or $R - mod$ finite simple objects are geometrically finite and hence in this cases $\zeta_g = \zeta$. But this is not the case of the category of modules over a Λ -ring, as we will see below.

Our aim is to apply the described above strategy for calculating ζ and ζ_g functions for the category of Λ -modules over F_1 . We know what is the polynomial ring over a Λ -ring, so we can use affine line $F_1[x]$ in our constructions. We show below that the categorical ζ -function of F_1 agrees with the results of chapter 3, where we calculated it via the algebraic closure and the Frobenius action.

First we should describe the category of modules over F_1 . This is done in full details in [H, chapter 2] for any Λ -ring. The constructions uses the functor W from unital commutative rings to Λ -rings which takes any ring R to its ring of Witt vectors $W(R)$. Originally the functor W was defined for rings with multiplicative unit. But the universal polynomials which define addition, multiplication and opposite in $W(R)$ do not use multiplicative unit so using the same formulas one can define the value of W on the non-unital rings.

If R is a Λ -ring then it comes with the Λ -ring map $\lambda_R : R \rightarrow W(R)$ which is defined by lambda operations on R . More precisely, if $\Lambda(R)$ denote the ring of invertible formal power series over R then $\lambda_R = E \circ \lambda_t$ where E is the Artin-Hasse exponential isomorphism of $\Lambda(R)$ and $W(R)$ (see [Y, chapter 4]) and

$$\lambda_t(r) = \sum_{i=0}^{\infty} \lambda^i(r) t^i$$

As it is proved in [H] the category of modules over a Λ -ring R , by which we mean the category $(Ring^\lambda/R)^+$ of abelian group objects in $Ring^\lambda/R$, is equivalent to the category $R - mod^\lambda$ of Λ -modules over R . A Λ -module over R is an R module M with a map $\lambda_M : M \rightarrow W(M)$ which is equivariant with respect to the Λ -structure of R . Here $W(X)$ denotes the Witt ring construction applied to the non-unital ring X with trivial multiplication. It is easy to check that in this case $W(M)$ has also trivial multiplication and additively is equal to the infinite product of M . It is shown in [H, Lemma 2.2] that we have an isomorphism of rings

$$i : W(R) \triangleright W(M) \rightarrow W(R \triangleright M)$$

which is induced by the canonical inclusions of R and M into $R \triangleright M$. A Λ -module M corresponds in the equivalence of $(Ring^\lambda/R)^+$ and $R - mod^\lambda$ to the Λ -ring $R \triangleright M$ with the Λ -ring structure defined by the composition

$$R \triangleright M \xrightarrow{\lambda_R \oplus \lambda_M} W(R) \triangleright W(M) \xrightarrow{i} W(R \triangleright M)$$

We have another description of the category $R - \text{mod}^\lambda$ (see [H, Remark 2.6]). If M is an object of this category and $\lambda_M : M \rightarrow W(M)$ is a structural map then it has components $\lambda_{M,n} : M \rightarrow M$ because as sets $W(M) = \prod_N M$. Easy calculation shows $\lambda_{M,n}$ is $\psi_{R,n}$ equivariant, where $\psi_{R,n}$ is the n th Adams operation of R . This gives us description of the category $R - \text{mod}^\lambda$ as a category of left modules over a twisted monoid algebra $R^\psi[N]$ where the multiplicative monoid N acts on any object M through the maps $\lambda_{M,n}$.

With the understanding of the category $R - \text{mod}^\lambda$ presented above we can come back to our situation and analyze the category $F_1 - \text{mod}^\lambda$. Observe that the Newton formula which relates Adams and λ -operations implies $\psi_{F_1,n} = \text{id}$ for any natural n . This implies that $\lambda_{M,1} = \text{id}$ and for $n > 1$, $\lambda_{M,n} : M \rightarrow M$ is any (additive) group homomorphism. So we have:

Lemma 4.3: Every object (M, λ_M) in $F_1 - \text{mod}^\lambda$ consists of an abelian group M and a sequence of group homomorphisms $\lambda_{M,n} : M \rightarrow M$ satisfying $\lambda_{M,n} \circ \lambda_{M,m} = \lambda_{M,mn}$ and $\lambda_{M,1} = \text{id}$. Morphisms $(M, \lambda_M) \rightarrow (P, \lambda_P)$ are given as group homomorphisms $f : M \rightarrow P$ which satisfy $f \circ \lambda_{M,n} = \lambda_{P,n} \circ f$ for any natural n . An object (M, λ_M) is simple and finite if M is a cyclic group of prime order.

Proof. The description of $F_1 - \text{mod}^\lambda$ was achieved before the statement of the lemma. Observe that an endomorphism $f : M \rightarrow M$ of a cyclic group satisfies: for any subgroup $P < M$, $f(P) \subset P$. This implies immediately, that simple objects are as described.

Lemma 4.4: Every non-degenerate object in $F_1 - \text{mod}^\lambda$ is of the form (C_p, λ_{C_p}) , where $\lambda_{C_p,n} = 0$ for $n > 1$.

Proof. First of all, every non-degenerate object (M, λ_M) in $F_1 - \text{mod}^\lambda$ has to be finite and simple. Hence it is of the form (C_p, λ_{C_p}) . But (M, λ_M) is non-degenerate if additionally we have

$$1 < n(\mathbf{Z} \triangleright M) = |\text{Hom}_{\text{Rings}/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)| < \infty.$$

So we have to check when the number of Λ -ring maps $\mathbf{Z}[x] \rightarrow \mathbf{Z} \triangleright M$ over F_1 is finite but bigger 1 (we always have a zero morphism), where (M, λ_M) is of the form (C_p, λ_{C_p}) . Observe that if $\varphi \in \text{Hom}_{\text{Rings}/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)$ then $\varphi(x) = (0, m)$ for a certain $m \in M$. Because φ is a Λ -ring homomorphism it has to commute with Λ -operations on the source and the target. Recall that in $\mathbf{Z}[x]$, $\lambda^n(x) = 0$ for $n > 1$. Hence for $n > 1$ we calculate

4.1.1

$$0 = \varphi(\lambda^n(x)) = \lambda^n(\varphi(x)) = \lambda^n((0, m)).$$

It means that we are looking for such objects (C_p, λ_{C_p}) which give us vanishing of higher λ -operations on elements $(0, m) \in \mathbf{Z} \triangleright C_p$. Observe that in general λ_M is related to the Λ -operations on $\mathbf{Z} \triangleright M$ via the Artin-Hasse isomorphism so we have to check that formulas $\lambda^n((0, m)) = 0$ imply $\lambda_{C_p,n} = 0$ for $n > 1$.

We are going to use now the calculation from [H, Addendum 2.3], where the relation between the sequence $\lambda_{M,n}$ and $\lambda_{R \triangleright M}$ is calculated in full generality. We get that

$$\lambda_{\mathbf{Z} \triangleright C_p}((0, m)) = ((0, \lambda_{C_p,1}(m)), (0, \lambda_{C_p,2}(m)), (0, \lambda_{C_p,3}(m)), \dots)$$

Now we can use the general observation about the Artin-Hasse invariant. If $f(t) = 1 + \sum a_i t^i \in \Lambda(R)$ then we write $f(t) = \prod (1 - (-1)^i b_i t^i)$ and the Artin-Hasse isomorphism $E : \Lambda(R) \rightarrow W(R)$ takes f to the sequence (b_1, b_2, b_3, \dots) . Observe that if

$$(*) \quad b_i \cdot b_j = 0 \text{ for any } i \text{ and } j$$

then up to sign $(b_1, b_2, b_3, \dots) = (a_1, a_2, a_3, \dots)$.

Coming back to 4.4.1 we get that up to sign:

$$0 = \lambda^n((0, m)) = (0, \lambda_{C_p,n}(m))$$

This implies that in our case $\lambda_{C_p,n} = 0$ for $n > 1$, as desired.

Corollary 4.5: Recall that ζ_R denotes the Riemann ζ -function of integers. We have

$$\zeta_R = \zeta_g(s, F_1 - \text{mod}^\lambda)$$

FINAL REMARK. It seems that the success of the Deligne's approach to Weil conjectures was caused by the fact that for an affine variety over a finite field we have two ways of calculating the ζ function. One is classical (categorical) via looking at finite simple objects in corresponding category. The second is via algebraic closure, rich geometric structure and calculation of the same function via counting fixed points of the Frobenius action. In the present paper we tried to justify the statement that in the category of Λ -rings the same two approaches should work.

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Instytut Matematyki, University of Warsaw
ul.Banacha 2, 02-097 Warsaw, Poland
e-mail: betley@mimuw.edu.pl